Bounds on the bipartite entanglement entropy for oscillator systems with or without disorder

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Abstract

We give a direct alternative proof of an area law for the entanglement entropy of the ground state of disordered oscillator systems—a result due to Nachtergaele, Sims and Stolz [1]. Instead of studying the logarithmic negativity, we invoke the explicit formula for the entanglement entropy of Gaussian states to derive the upper bound. We also contrast this area law in the disordered case with divergent lower bounds on the entanglement entropy of the ground state of one–dimensional ordered oscillator chains.

The Model

Let $\mathbb{G} = (\mathcal{V}, \mathcal{E})$ be a connected, infinite graph with uniformly bounded degree $\mathcal{N} \in \mathbb{N}$. It is an easy exercise to establish that the latter property is equivalent to the exisistence of a $\mu > 0$ such that

$$\sup_{x \in \mathcal{V}} \sum_{y \in \mathcal{V}} e^{-\mu d(x,y)} < \infty, \tag{1}$$

where $d(\cdot, \cdot)$ denotes the natural graph distance on $\mathbb{G}$. Further, we pick two families of real–valued random variables $\{h^{(q)}_{xy}\}_{x,y \in \mathcal{V}}$ and $\{h^{(p)}_{xy}\}_{x,y \in \mathcal{V}}$. Suppose there is an exhaustive sequence of *finite* sets $\Lambda_n \nearrow \mathcal{V}$ such that the matrices $h^{(\sharp)}_{\Lambda_n} = (h^{(\sharp)}_{xy})_{x,y \in \Lambda_n}$, $\sharp \in \{p, q\}$, are almost surely symmetric and positive definite. Moreover, we assume the uniform norm bound

$$\sup_{n \in \mathbb{N}} \max \left\{ \|h^{(p)}_{\Lambda_n}\|, \|(h^{(p)}_{\Lambda_n})^{-1}\|, \|h^{(q)}_{\Lambda_n}\| \right\} \leq C \quad \text{a.s.}$$

for some deterministic $C < \infty$. On a suitably chosen dense domain within the Hilbert space $\mathcal{H}_n = \bigotimes_{x \in \Lambda_n} L^2(\mathbb{R}, dq_x)$ we install the usual self–adjoint position and momentum operators $q = (q_x)_{x \in \Lambda_n}$ and $p = (p_x)_{x \in \Lambda_n}$, respectively. These operators constitute the *finite volume Hamiltonian*

$$H_{\Lambda_n} = (q^T \; p^T) \begin{pmatrix} h^{(q)}_{\Lambda_n} & 0 \\ 0 & h^{(p)}_{\Lambda_n} \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix}. \tag{2}$$

Entanglement Entropy of Gaussian States

Let $\Lambda_0 \subset \mathcal{V}$ be a finite distinguished region and assume $n \in \mathbb{N}$ large enough such that $\Lambda_0 \subset \Lambda_n$. Then for a density operator $\sigma_{\Lambda_n} \in \mathcal{B}(\mathcal{H}_n)$, the *entanglement entropy* with respect to the bipartition

$$\mathcal{H}_n = \bigotimes_{x \in \Lambda_0} L^2(\mathbb{R}, dq_x) \otimes \bigotimes_{x \in \Lambda_n \setminus \Lambda_0} L^2(\mathbb{R}, dq_x) \tag{3}$$

is given by

$$S(\sigma_{\Lambda_n}; \Lambda_0) = -\operatorname{tr}(\sigma_{\Lambda_0} \log \sigma_{\Lambda_0}),$$

where $\sigma_{\Lambda_0} = \operatorname{tr}_{\Lambda_n \setminus \Lambda_0}(\sigma_{\Lambda_n})$.
It is a well–known fact that the unique ground state of the finite volume Hamiltonian (2) is Gaussian and can be represented by a density matrix, which we shall denote by ρ_{Λ_0}. Introducing the shorthand $r = q \oplus p$, its entanglement entropy with respect to the bipartition (3) can be computed in terms of the *covariance matrix* Γ,

$$\Gamma_{xy} = \operatorname{tr}(\rho_{\Lambda_n}\{r_x, r_y\}), \qquad x, y = 1, \ldots, 2|\Lambda_n|,$$

with the help of the handy formula

$$S(\rho_{\Lambda_n}; \Lambda_0) = \sum_{\gamma \in \sigma_{\mathsf{symp}}(\tilde{\Gamma})} \left(\frac{\gamma}{2} + \frac{1}{2}\right) \log\left(\frac{\gamma}{2} + \frac{1}{2}\right) - \left(\frac{\gamma}{2} - \frac{1}{2}\right) \log\left(\frac{\gamma}{2} - \frac{1}{2}\right), \tag{4}$$

where $\tilde{\Gamma}$ is obtained from Γ by erasing the rows and columns belonging to $\Lambda_n \setminus \Lambda_0$. Here, $\sigma_{\mathsf{symp}}(\cdot)$ denotes the symplectic spectrum counting multiplicities, which is in turn characterized by the following theorem due to Williamson [3]:

Theorem *Let $\Gamma \in \mathbb{R}^{2n \times 2n}$ be symmetric and positive definite and define a symplectic form on $\mathbb{R}^{2n \times 2n}$ by*

$$\Omega = \begin{pmatrix} 0 & -\mathbb{1}_n \\ \mathbb{1}_n & 0 \end{pmatrix}.$$

Then there exists a symplectic matrix $S \in \mathsf{SP}(2n, \mathbb{R}) = \{S \in \mathbb{R}^{2n \times 2n} \,|\, S^T \Omega S = \Omega\}$ such that

$$S^T \Gamma S = \begin{pmatrix} \mathcal{L} & 0 \\ 0 & \mathcal{L} \end{pmatrix},$$

where $\mathcal{L} = \operatorname{diag}(\gamma_1, \ldots, \gamma_n) > 0$. The numbers $\gamma_1, \ldots, \gamma_n$, which are called symplectic eigenvalues *of Γ and which form the* symplectic spectrum *$\sigma_{\mathsf{symp}}(\Gamma) = \{\gamma_k\}_{k=1}^n$, can be computed as the positive eigenvalues of $i\Gamma^{1/2}\Omega\Gamma^{1/2}$ or as the imaginary part of the eigenvalues of $\Gamma\Omega$.*

As an atavism of Heisenberg's celebrated uncertainty relation, the symplectic eigenvalues of the covariance matrices Γ and $\tilde{\Gamma}$ lay in the interval $[1, \infty)$ so that formula (4) is well defined.

Furthermore, it is—at least since the work of Cirac, Schuch, and Wolf [2]—well known that for a finite volume Hamiltonian of the form (2) the covariance matrix of the ground state takes a particularly simple form:

$$\Gamma = \begin{pmatrix} (h^{(p)}_{\Lambda_n})^{1/2} h^{-1/2}_{\Lambda_n} (h^{(p)}_{\Lambda_n})^{1/2} & 0 \\ 0 & (h^{(p)}_{\Lambda_n})^{-1/2} h^{1/2}_{\Lambda_n} (h^{(p)}_{\Lambda_n})^{-1/2} \end{pmatrix}, \quad h_{\Lambda_n} = (h^{(p)}_{\Lambda_n})^{1/2} h^{(q)}_{\Lambda_n} (h^{(p)}_{\Lambda_n})^{1/2}. \tag{5}$$

Main Results

Invoking the explicit formulae (4) and (5), we give an alternative proof of a result due to Nachtergaele, Sims, and Stolz [1] asserting an area law bound on the entanglement under a suitable localization assumption:

Theorem. *Concomitant to the assumptions above, suppose that there exist $c < \infty$ and $\nu \in (2 \log \mathcal{N}, \infty)$ such that*

$$\mathbb{E}\left[\left|\left\langle (h^{(p)}_{\Lambda_n})^{1/2} \delta_x, h^{-1/2}_{\Lambda_n} (h^{(p)}_{\Lambda_n})^{1/2} \delta_y \right\rangle\right|\right] \leq c\, e^{-\nu d(x,y)} \tag{6}$$

for all $n \in \mathbb{N}$. Then, there exists $C \in (0, \infty)$ such that for any finite subset $\Lambda_0 \subset \mathcal{V}$

$$\mathbb{E}\left[S(\rho_{\Lambda_n}; \Lambda_0)\right] \leq C\, |\partial \Lambda_0|$$

for all $n \in \mathbb{N}$ with $\Lambda_0 \subset \Lambda_n$.

In view of this theorem, it is natural to ask whether the area law behavior of the entanglement entropy is violated when dropping the localization assumption (6). This questions is answered positively as we illustrate by the investigation of the one–dimensional toy system depicted in Figure 1 below. It is defined by choosing $\mathcal{V} \in \{\mathbb{N}, \mathbb{Z}\}$,

$$h^{(q)}_{\Lambda_n} = \begin{pmatrix} 2 & -1 & & 0 \\ -1 & 2 & \ddots & \\ & \ddots & \ddots & -1 \\ 0 & & -1 & 2 \end{pmatrix}, \quad \text{and } h^{(p)}_{\Lambda_n} = \mathbb{1} \tag{7}$$

in the finite volume Hamiltonian (2).

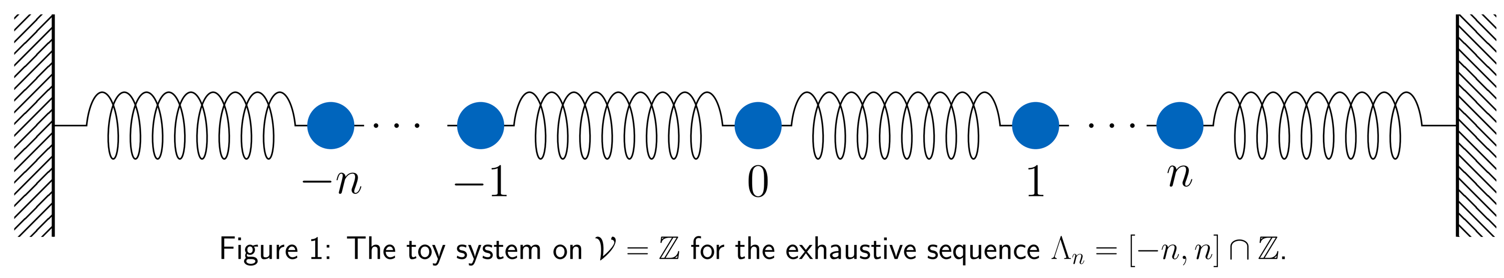

Figure 1: The toy system on $\mathcal{V} = \mathbb{Z}$ for the exhaustive sequence $\Lambda_n = [-n, n] \cap \mathbb{Z}$.

We can prove the following theorem for this system:

Theorem *Let $\mathcal{V} \in \{\mathbb{N}, \mathbb{Z}\}$ with next-neighbor edges. Then there exist exhaustive sequences $(\Lambda_n)_{n \in \mathbb{N}}, (\Lambda_0^{(n)})_{n \in \mathbb{N}}$ of finite sets with $\Lambda_0^{(n)} \subset \Lambda_n \subset \mathcal{V}$ for all $n \in \mathbb{N}$ such that for the sequence of ground state density matrices $(\rho_{\Lambda_n})_{n \in \mathbb{N}}$ of the finite volume Hamiltonians specified by (7),*

$$\lim_{n \to \infty} S(\rho_{\Lambda^{(n)}}; \Lambda_0^{(n)}) = \infty.$$

Idea of the Proof of the Violation of an Area Law

In either case $\mathcal{V} \in \{\mathbb{N}, \mathbb{Z}\}$, we exploit the fact that the spectral decomposition of the tridiagonal matrix $h^{(q)}_{\Lambda_n}$ is known explicitly and that the resulting sums can be well aproximated by Riemann integrals for large $|\Lambda_n|$.

$\mathcal{V} = \mathbb{N}$: Here, we solve the aforementioned integrals and establish a divergent lower bound on the last diagonal entry of the matrix product whose ordinary spectrum constitutes the entanglement entropy in light of (4) and Williamson's theorem. The min–max theorem then implies the divergence of the largest eigenvalue, and hence, the divergence of the largest symplectic eigenvalue.

$\mathcal{V} = \mathbb{Z}$: Due to the emergence of translation invariance of the system in the limit $\Lambda_n \nearrow \mathbb{Z}$, the limiting operators are Toeplitz. Hence, after deriving a lower bound on the entanglement entropy (4) in terms of determinants of Toeplitz matrices, we have a tremendous arsenal of powerful machinery at our disposal. In particular, we appeal to Szegő's strong limit theorem to establish the divergence of the determinants in the lower bound.

Eventually, a thorough error analysis furnishes explicit exhaustive sequences of finite sets for which we may infer the asserted divergence of the entanglement entropy.

References

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